ON THE FOURIER TRANSFORMS OF AN INTERESTING CLASS OF MEASURES

BY
J. R. BLUM* AND BERNARD EPSTEIN**

ABSTRACT

Let $\{X_k, k = 1,2,\ldots\}$ be a sequence of independent binomial variables, with $P\{X_k = 1\} = 1 - P\{X_k = 0\} = p_k$ · Let $Y = \sum_{k=1}^{\infty} X_k/2^k$ and $\hat{\mu}(t)$ be the Fourier transform of the distribution of Y. Finally denote $\overline{\lim}_{k\to\infty} [P_k - 1/2]$ by δ. We have

Theorem. $(4/\pi)\delta \leqq \overline{\lim}_{t\to\infty} |\hat{\mu}(t)| \leqq 2\delta.$

1. Introduction

Let $\{X_k,\ k = 1, 2, \cdots\}$ be a sequence of independent random variables, X_k assuming the values 0 and 1 with probabilities q_k and p_k $(= 1 - q_k)$ respectively. Let μ be the measure on the unit interval associated with the random variable

$$Y = \sum_{k=1}^{\infty} \frac{X_k}{2^k}.$$

The class of all measures that can be obtained in this manner will be denoted ϕ. Marsaglia [1] has shown that ϕ contains a large subclass of continuous measures which are singular with respect to Lebesgue measure (which is, of course, that member of ϕ corresponding to the values $p_1 = p_2 = \cdots = \frac{1}{2}$).

Let $\delta = \overline{\lim}_{k\to\infty} \left| p_k - \frac{1}{2} \right|$ and let $\hat{\mu}(t)$ be the Fourier-Stieltjes transform of μ, defined for all real t as $\int \exp(2\pi i t x) d\mu(x)$. (Since $\hat{\mu}(-t) = \overline{\hat{\mu}(t)}$, we may confine attention to non-negative t.) In this note the following result is proven.

* Research supported by N.S.F. Grant GP–25736.
** Research supported by N.S.F. Grant GP–12365.
Received April 15, 1971

THEOREM. $(4/\pi)\delta \leqq \overline{\lim}_{t\to\infty} |\hat{\mu}(t)| \leqq 2\delta.$

As an immediate corollary it follows that $\hat{\mu}(t)$ vanishes at infinity if and only if $\delta = 0$, which is equivalent to the condition $p_k \to \frac{1}{2}$. Furthermore, from the well-known fact [2] that $\hat{\mu}(t)$ vanishes at infinity if (and only if) the Fourier-Stieltjes coefficients $\hat{\mu}(n)$ $(n = 1, 2, \cdots)$ approach zero as n increases, it follows that these coefficients approach zero if and only if $p_k \to \frac{1}{2}$.

PROOF. Since the random variables X_k are independent, it follows that

$$\hat{\mu}(t) = \prod_{k=1}^{\infty} (q_k + p_k \exp 2\pi i t/2^k).$$

By a simple calculation one obtains

$$(1) \qquad |\hat{\mu}(t)|^2 = \prod_{k=1}^{\infty} \left(\cos^2 \frac{\pi t}{2^k} + 4\varepsilon_k^2 \sin^2 \frac{\pi t}{2^k}\right), \quad \varepsilon_k = p_k - \tfrac{1}{2}.$$

Now suppose that $\delta > 0$. Let δ' be any positive number less than δ. There exists an increasing sequence $k_1, k_2, \cdots$ such that $|\varepsilon_{k_i}| > \delta'$. For each index i let $t_i = 2^{k_i - 1}$. Then $\cos^2 \pi t_i/2^k = 1$ and $\sin^2 \pi t_i/2^k = 0$ for $k < k_i$, and so the first $k_i - 1$ factors in (1) assume the value 1. For $k = k_i$ the factor assumes the value $4\varepsilon_{k_i}^2$, which exceeds $4\delta'^2$, and the remaining factors are at least as large as $\cos^2 \pi/2^{k-k_i+1}$. Thus

$$|\hat{\mu}(2^{k_i-1})|^2 \geqq 4\delta'^2 \prod_{k=k_i+1} \cos^2 \frac{\pi}{2^{k-k_i+1}}$$

$$= 4\delta'^2 \prod_{i=2}^{\infty} \cos^2 \frac{\pi}{2^i} = 4\delta'^2 \cdot \frac{4}{\pi^2} = \frac{16\delta'^2}{\pi^2}.$$

(The evaluation of the infinite product is obtained from the identity $\prod_{r=1}^{\infty} \cos u/2^r = \sin u/u$, which will be used again later.) Thus, $\overline{\lim}_{t\to\infty} |\hat{\mu}(t)| \geqq 4\delta'/\pi$, and since δ' may be chosen arbitrarily close to δ, it follows that $\overline{\lim}_{t\to\infty} |\hat{\mu}(t)| \geqq 4\delta/\pi$. Since this inequality is trivially true when $\delta = 0$, the first half of the theorem is proved.

When $\delta = \frac{1}{2}$ the second inequality is certainly true, for $|\hat{\mu}(t)|$, and hence $\overline{\lim}_{t\to\infty} |\hat{\mu}(t)|$, cannot exceed unity. Furthermore, when $|\varepsilon_k| = \frac{1}{2}$ for all indices

k the identity $|\hat{\mu}(t)| \equiv 1$ holds, and so the factor 2 which multiplies δ is best possible (although for values of δ less than $\frac{1}{2}$ it may be possible that a smaller factor *depending on* δ is correct).

Now, for any value of δ less than $\frac{1}{2}$ let α be any number satisfying $4\delta^2 < \alpha \leqq 1$. Then an integer M may be chosen such that $4\varepsilon_k^2 < \alpha$ whenever k exceeds M, and since no factor appearing in (1) can exceed unity, it follows that

$$
\begin{aligned}
|\hat{\mu}(t)|^2 &\leqq \prod_{k=M+1}^{\infty} \left(\cos^2 \frac{\pi t}{2^k} + \alpha \sin^2 \frac{\pi t}{2^k}\right) \\
&= \prod_{k=1}^{\infty} \left(\cos^2 \frac{\pi s}{2^k} + \alpha \sin^2 \frac{\pi s}{2^k}\right), \qquad s = 2^{-M} t.
\end{aligned} \tag{2}
$$

Since each term appearing in each factor of the second product is non-negative, the value of this product, which is henceforth denoted $f(s,\alpha)$, is unaffected by multiplying out the factors and assembling them according to powers of α. We obtain

$$
\alpha^{-1} f(s,\alpha) = \alpha^{-1} \prod_{k=1}^{\infty} \cos^2 \frac{\pi s}{2^k} + f_1(s) + f_2(s)\alpha + \cdots, \tag{3}
$$

where the exact nature of the functions $f_1(s), f_2(s), \cdots$ is of no concern. Since $\prod_{k=1}^{\infty} \cos^2 \pi s/2^k = \sin^2 \pi s/\pi^2 s^2$, which vanishes at infinity, it follows that

$$
\alpha^{-1} \overline{\lim_{s \to \infty}} f(s,\alpha) = \overline{\lim_{s \to \infty}} \{f_1(s) + f_2(s)\alpha + \cdots\}. \tag{4}
$$

Since each of the functions $f_1, f_2, \cdots$ is non-negative, the sum appearing on the right side, and hence its upper limit, is not decreased when α is replaced by unity; when this substitution is performed the sum clearly becomes $f(s,1) - \sin^2 \pi s/\pi^2 s^2$, and since $f(s,1)$ reduces identically to unity, we obtain from (4) the inequality $\alpha^{-1} \overline{\lim}_{s \to \infty} f(s,\alpha) \leqq 1$, or $\lim_{s \to \infty} f(s,\alpha) \leqq \alpha$. Hence $\overline{\lim}_{t \to \infty} |\hat{\mu}(t)|^2 \leqq 4\delta^2$, which is equivalent to the desired result.

While the factor 2 appearing in the statement of the theorem is, as explained above, best possible, it appears rather likely that the other factor, namely $4/\pi$, is not, for it was obtained by estimating $|\hat{\mu}(t)|$ on a very "thin" sequence of values of t, namely certain powers of 2.

According to one of Marsaglia's results presented in [1], the measure μ will be singular (with respect to Lebesgue measure) if and only if the series $\sum_{k=1}^{\infty} \varepsilon_k^2$ diverges. It now follows immediately from the result established in the present note that the measure determined by the probabilities $p_k = \frac{1}{2}(1 + k^{-\frac{1}{2}})$ is singular but possesses a Fourier transform vanishing at infinity.

REFERENCES

1. G. Marsaglia, *Random variables with independent binary digits*, submitted to Ann. Math. Statist.

2. R. Salem, *Algebraic Numbers and Fourier Analysis*, p. 38. D. C. Heath and Co., Boston, 1963.

THE UNIVERSITY OF NEW MEXICO